Brownian motion of a charged test particle near a reflecting boundary at finite temperature

Hongwei Yu, Jun Chen and Puxun Wu

Department of Physics and Institute of Physics,

Hunan Normal University, Changsha, Hunan 410081, China

Abstract

We discuss the random motion of charged test particles driven by quantum electromagnetic fluctuations at finite temperature in both the unbounded flat space and flat spacetime with a reflecting boundary and calculate the mean squared fluctuations in the velocity and position of the test particle. We show that typically the random motion driven by the quantum fluctuations is one order of magnitude less significant than that driven by thermal noise in the unbounded flat space. However, in the flat space with a reflecting plane boundary, the random motion of quantum origin can become much more significant than that of thermal origin at very low temperature.

PACS numbers:

I. INTRODUCTION

Quantum fluctuations, especially quantum vacuum fluctuations, have been subjected to extensive studies, since the emergence of quantum theory which has profoundly changed our conception of empty space or vacuum. Two well-known examples of experimentally verified effects resulting from changes of vacuum fluctuations are the Lamb shift and the Casimir effect [1, 2, 3]. A fundamental feature to be expected of any field which is quantized is the quantum fluctuations. Therefore, test particles under the influence of these quantum field fluctuations will no longer move on the classical trajectories, but undergo random motion around a mean path. It will be very desirable and quite interesting to bring to light the basic features of this kind of random motion driven by quantum, as opposed to classical or thermal-like fluctuations.

In investigating the random motion of test particles driven by quantum field fluctuations, a natural first step is to examine the case of vacuum, since, quantum-theoretically, quantum fields fluctuate even in vacuum. However, because of the divergences that arise in quantum field theory in unbounded Minkowski spacetime when vacuum is concerned, it appears that the most tractable cases of random motion of test particles in vacuum would be those in which changes of vacuum fluctuations occur due to the presence of boundaries or non-trivial topology in a local flat spacetime. The simplest example of this is the random motion of a charged test particle caused by *changes* in the electromagnetic vacuum fluctuations near a perfectly reflecting plane boundary, which has recently been investigated [4][1]. There, the effects have been calculated of the modified electromagnetic vacuum fluctuations due to the presence of the boundary upon the motion of a charged test particle . In particular, it has been shown [4] that the mean squared fluctuations in velocity and position of the test particle normal to the plane can be associated with an effective temperature of

$$T_{eff} = \frac{\alpha}{\pi}\,\frac{1}{k_B m z^2} = 1.7 \times 10^{-6}\left(\frac{1\mu m}{z}\right)^2\ K = 1.7 \times 10^{2}\left(\frac{1\AA}{z}\right)^2\ K\ , \qquad (1)$$

where k_B is Boltzmann's constant and z is the distance from the boundary. This might be experimentally accessible in the future. These results have also been generalized to the case of two parallel reflecting plates [9].

As further step along the line, naturally, one would be interested in a physically more interesting case, i.e., the random motion of test particles caused by quantum field fluctuations at non-zero temperature (as opposed to zero temperature vacuum fluctuations) in the unbounded flat spacetime and flat spacetimes with boundaries. These are just what we want to address in the present paper. We would like to

[1] Another example of this quantum random motion is the random motion of photons due to modified quantum fluctuations of the quantized gravitational field [5, 6, 7, 8], which induces quantum lightcone fluctuations.

study the random motion of a charged test particle subject to ever-existing quantum electromagnetic fluctuations at finite temperature, i.e., the random motion driven by quantum fluctuations of a thermal bath of photons. It will be demonstrated that, for the random motion driven by quantum electromagnetic field fluctuations at finite temperature, no dissipation is needed for the velocity dispersion of the test particle to be bounded at later times, in contrast to that driven by thermal noise. Moreover, it will be shown that, in the unbounded flat spacetime, generally the random motion driven by quantum fluctuations is one order of magnitude less significant than that driven by thermal noise. However, it could be strengthened if the quantum field fluctuations are to be modified by the presence of a reflecting plane boundary and even become orders of magnitude more significant than that of thermal origin, when the system temperature is low.

II. BROWNIAN MOTION OF THE TEST PARTICLE IN MINKOWSKI SPACE AT FINITE TEMPERATURE

First, let us now consider the motion of a charged test particle subject to quantum electromagnetic field fluctuations at finite temperature T in the Minkowski (unbounded flat) space. We will use Lorentz-Heaviside units with $c = \hbar = 1$ in our discussions. In the limit of small velocities, the motion of a charged particle is described by a non-relativistic equation of motion (Langevin equation) with a fluctuating electric force

$$\frac{d\mathbf{v}}{dt} = \frac{e}{m}\,\mathbf{E}(\mathbf{x},t)\,; \tag{2}$$

assuming that the particle is initially at rest and has a charge to mass ratio of e/m. The velocity of the charged particle at time t can be calculated as follows

$$\mathbf{v} = \frac{e}{m}\int_0^t \mathbf{E}(\mathbf{x}\,,t)\;dt = \left(\frac{4\pi\alpha}{m^2}\right)^{1/2}\int_0^t \mathbf{E}(\mathbf{x}\,,t)\;dt\;, \tag{3}$$

where α is the fine-structure constant. The mean squared fluctuations in speed in the i-direction can be written as (no sum on i)

$$\langle \Delta v_i^2\rangle \;=\; \frac{4\pi\alpha}{m^2}\int_0^t\int_0^t \langle E_i(\mathbf{x},t_1)\;E_i(\mathbf{x},t_2)\rangle_\beta\, dt_1 dt_2\;, \tag{4}$$

where $\langle E_i(\mathbf{x},t_1)\;E_i(\mathbf{x},t_2)\rangle_\beta$ is understood to be the renormalized electric field two-point function at finite temperature $T = \frac{1}{k_B\beta}$ and we have used the fact that $\langle E_i\rangle_\beta = 0$. We adopt the well-established renormalization procedure in quantum field theory in which physical quantities are calculated and supposedly experimentally measured against vacuum. Therefore, the renormalized electric field two-point function is obtained by subtracting the vacuum contribution. We have, for simplicity, assumed that the distance does not change significantly on the time scale of interest in a time t, so that it can be treated approximately as a constant. If there is

a classical, nonfluctuating field in addition to the fluctuating quantum field, then Eq. (4) describes the velocity fluctuations around the mean trajectory caused by the classical field. Note that when the initial velocity does not vanishes, one has to also consider the influence of fluctuating magnetic fields on the velocity dispersion of the test particles. However, it has been shown that this influence is, in general, of the higher order than that caused by fluctuating electric fields and is thus negligible [11].

Let us note that the two point function for the photon field at finite temperature, $D^{\mu\nu}_{\beta}(x,x') = \langle 0|A^{\mu}(x)\, A^{\nu}(x')|0\rangle_{\beta}$, can be written as an infinite imaginary-time image sum of the corresponding zero-temperature two-point function, $D^{\mu\nu}_{0}(x-x')$, i.e.,

$$D^{\mu\nu}_{\beta}(x,x') = \sum_{n=-\infty}^{\infty} D^{\mu\nu}_{0}(\mathbf{x}-\mathbf{x}', t-t'+in\beta)\ , \tag{5}$$

where argument x stands for a four-vector, i.e., ($\mathbf{x}, t$). In the Feynman gauge, we have

$$D^{\mu\nu}_{0}(x-x') = \frac{\eta^{\mu\nu}}{4\pi^2(\Delta t^2 - \Delta \mathbf{x}^2)}\ . \tag{6}$$

By taking the four dimensional curl in x and in x', we can obtain the electric field two-point function from that of the photon field as follows

$$\langle E_i(x)E_j(x')\rangle = \langle F_{0i}(x)F_{0j}(x')\rangle = \partial_0\partial'_0\langle A_i(x)A_j(x')\rangle + \partial_i\partial'_j\langle A_0(x)A_0(x')\rangle\ . \tag{7}$$

The components of the renormalized electric field two-point function at finite temperature, $\langle \mathbf{E}(\mathbf{x},t_1)\,\mathbf{E}(\mathbf{x},t_2)\rangle_{\beta}$, can be obtained by taking curl of Eq. (5) according to Eq. (7) and dropping the vacuum term ($n=0$ term in the sum). The result is

$$\begin{aligned}&\langle E_x(\mathbf{x},t')E_x(\mathbf{x},t'')\rangle_{\beta} = \langle E_y(\mathbf{x},t')E_y(\mathbf{x},t'')\rangle_{\beta} = \langle E_z(\mathbf{x},t')E_z(\mathbf{x},t'')\rangle_{\beta} \\ &= \frac{1}{\pi^2}\sum_{n=-\infty}^{\infty}{}' \frac{1}{(\Delta t + in\beta)^4} = \frac{\pi^2}{3\beta^4}\left(2+\cosh\frac{2\pi\Delta t}{\beta}\right)\mathrm{csch}^4\left(\frac{\pi\Delta t}{\beta}\right) - \frac{1}{\pi^2\Delta t^4}\ .\end{aligned} \tag{8}$$

Here a prime means that the $n=0$ term is omitted in the summation. It is interesting to note that the first term in the above result is the usual finite temperature correlation function that satisfies the Kubo-Martin-Schwinger relation while the last is the vacuum term (zero temperature contribution). Therefore, the renormalized correlation function does not obey the KMS relation. Mathematically one can obtain a regularized correlation function that satisfies the KMS relation by subtracting both the $n=0$ and $n=1$ terms. However, the problem is that one does see any physical motivation in removing the $n=1$ mode in contrast to in deducting the $n=0$ one which amounts to taking away the vacuum contribution. Let us also note here that the two-point electromagnetic field correlation functions in black body radiation have been examined in the literature, see for example, Ref. [10, 12].

Substituting the above results into Eq. (4) and carrying out the integration, we find that the velocity dispersions are given by

$$\langle \Delta v_x^2\rangle \ = \ \langle \Delta v_y^2\rangle = \langle \Delta v_z^2\rangle = \frac{e^2}{m^2}\int_0^t\int_0^t \langle E_x(\mathbf{x},t')\ E_x(\mathbf{x},t'')\rangle_{\beta}\, dt'\, dt''$$

$$= \frac{e^2 \text{csch}^2(\frac{\pi t}{\beta})}{18\pi^2 m^2 \beta^2 t^2} \left[5\pi^2 t^2 + 3\beta^2 + (\pi^2 t^2 - 3\beta^2) \cosh \frac{2\pi t}{\beta} \right] . \tag{9}$$

In the low temperature limit, i.e., when $\beta \gg t$, we have

$$\langle \Delta v^2 \rangle = \langle \Delta v_x^2 \rangle + \langle \Delta v_y^2 \rangle + \langle \Delta v_z^2 \rangle = \frac{e^2 \pi^2}{15 m^2 \beta^2} \left(\frac{t}{\beta} \right)^2 - \frac{2 e^2 \pi^4}{189 m^2 \beta^2} \left(\frac{t}{\beta} \right)^4 . \tag{10}$$

This result shows that the velocity dispersion decreases very quickly as inverse powers of β^4 and it approaches zero when $\beta \to \infty$ as expected. While in the high temperature limit, i.e., when $t \gg \beta$,

$$\langle \Delta v^2 \rangle = \frac{e^2}{3 m^2 \beta^2} - \frac{e^2}{\pi^2 m^2 t^2} . \tag{11}$$

To get a concrete idea of how large t should be in order that the condition $t \gg \beta$ is fulfilled, let us assume that the temperature T is about $\sim 10^2$ Kevin, which can well be considered as high since we are discussing a quantum effect, then the condition becomes $t \gg 5.7 \times 10^{-14} sec.$. This is rather small. It is interesting to note that, for the random motion driven by quantum fluctuations at finite temperature here, no dissipation is needed for $\langle \Delta v_i^2 \rangle$ to be bounded at late times in contrast to the random motion due to thermal noise.

The mean squared position fluctuations can be calculated as follows

$$\begin{aligned} \langle \Delta x^2 \rangle &= \langle \Delta y^2 \rangle = \langle \Delta z^2 \rangle = \int_0^t dt_1 \int_0^{t_1} dt' \int_0^t dt_2 \int_0^{t_2} dt'' \, \langle E_x(\mathbf{x}, t') \, E_x(\mathbf{x}, t'') \rangle_\beta \\ &= \frac{e^2}{18\pi^2 m^2 \beta^2} \left(\pi^2 t^2 - 6\pi t \beta \coth \frac{\pi t}{\beta} + 6\beta^2 \left[1 + \ln \left(\frac{\beta}{\pi t} \sinh \frac{\pi t}{\beta} \right) \right] \right) . \end{aligned} \tag{12}$$

The limiting forms for both low and high temperature approximations are respectively

$$\langle \Delta x^2 \rangle = \frac{e^2 \pi^2 t^4}{180 m^2 \beta^4} - \frac{\pi^4 t^6}{1701 m^2 \beta^6} , \qquad \beta \gg t , \tag{13}$$

and

$$\langle \Delta x^2 \rangle = \frac{e^2 t^2}{18 m^2 \beta^2} - \frac{e^2}{3\pi^2 m^2} \ln \frac{\pi t}{\beta} + \frac{e^2}{3\pi^2 m^2} , \qquad t \gg \beta . \tag{14}$$

Eq. (14) reveals that $\sqrt{\langle \Delta x^2 \rangle}$ grows linearly with time, and thus in principle can increase indefinitely with time. However, recall that we have assumed that the particle do not move very far on the time scale of interest in a time t. Therefore, it is quite compelling for us to figure out under what conditions Eq. (14) is compatible with our initial approximation, which disregards the displacement of the particle. For this purpose, let us note that a natural time scale of interest here is set by β, the inverse of the temperature of the system. Hence, we expect our results to be a good approximation as long as $\langle \Delta x^2 \rangle \ll \beta^2$. This equivalent to requiring that

$$t \ll \frac{3}{\sqrt{2\alpha\pi}} (m\beta)\beta . \tag{15}$$

Note that $m\beta$ is the ratio of the temperature corresponding to the mass of the particle to that of the system, which is typically very large. Take an electron for example, the temperature corresponding to the electron mass is $\sim 5.93 \times 10^9$ K. Therefore, our results can be valid as long as the system temperature is not any close to this value. This is expected to be fulfilled by any experiment at the Earth. Finally, let us note that this kind of random motion driven by quantum fluctuations is superimposed on that driven by thermal noise. Let the root mean squared fluctuations in velocity due to the random motion driven by quantum fluctuations be denoted by$\Delta v_{qm} = \sqrt{\langle \Delta v^2 \rangle}$ and that by thermal noise at the same temperature by Δv_{th}, then it is easy to show that

$$\frac{\Delta v_{qm}}{\Delta v_{th}} = \frac{2}{3}(\pi\alpha)^{1/2} \approx 0.1 = 10^{-1} \ . \tag{16}$$

This indicates that typically the random motion driven by quantum fluctuations is one order of magnitude less significant than that driven by thermal noise.

III. BROWNIAN MOTION OF THE TEST PARTICLE NEAR A REFLECTING BOUNDARY AT FINITE TEMPERATURE

Now a question arises naturally as to what happens if we modify the quantum field fluctuations by adding a boundary in space, a perfectly reflecting plane, for example. In particular, we are interested in whether the random motion driven by quantum field fluctuations at finite temperature will be strengthened or weakened by the modification. Suppose such a reflecting plate be located at the $z = 0$ plane and the test particle be initially at a distance z from the plate, then the electric field two-point function at finite temperature, $\langle \mathbf{E}(\mathbf{x}, t_1)\ \mathbf{E}(\mathbf{x}, t_2)\rangle_\beta$, can be found by the method of double images with one involving an image source displaced in the z-direction and the other involving an infinite sum of temperature images displaced in imaginary time. At a point a distance z from the plane, the results are

$$\begin{aligned}
&\langle E_x(\mathbf{x}, t')E_x(\mathbf{x}, t'')\rangle_\beta = \langle E_y(\mathbf{x}, t')E_y(\mathbf{x}, t'')\rangle_\beta \\
&\quad = \sum_{n=-\infty}^{\infty}{}' \frac{1}{\pi^2(\Delta t + in\beta)^4)} - \sum_{n=-\infty}^{\infty} \frac{(\Delta t + in\beta)^2 + 4z^2}{\pi^2[(\Delta t + in\beta)^2 - 4z^2)]^3} \\
&\quad \equiv F_{\beta m}(\Delta t, z) + F^x_{\beta b}(\Delta t, z) \ ,
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\langle E_z(\mathbf{x}, t')E_z(\mathbf{x}, t'')\rangle_\beta \\
&\quad = \sum_{n=-\infty}^{\infty}{}' \frac{1}{\pi^2(\Delta t + in\beta)^4)} - \sum_{n=-\infty}^{\infty} \frac{1}{\pi^2[(\Delta t + in\beta)^2 - 4z^2)]^2} \\
&\quad \equiv F_{\beta m}(\Delta t, z) + F^z_{\beta b}(\Delta t, z) \ ,
\end{aligned} \tag{18}$$

where we have defined

$$F_{\beta m}(\Delta t, z) \;=\; \frac{\pi^2}{3\beta^4}(2+\cosh\frac{2\pi\Delta t}{\beta})\mathrm{csch}^4\Big(\frac{\pi\Delta t}{\beta}\Big) - \frac{1}{\pi^2\Delta t^4} \tag{19}$$

and

$$\begin{aligned} F^x_{\beta b}(\Delta t, z) \;=\; & -\frac{1}{64\pi\beta z^3}\Big(\coth\frac{\pi(\Delta t-2z)}{\beta} - \coth\frac{\pi(\Delta t+2z)}{\beta}\Big) \\ & +\frac{1}{32\beta^2 z^2}\Big(\mathrm{csch}^2\frac{\pi(\Delta t-2z)}{\beta} + \mathrm{csch}^2\frac{\pi(\Delta t+2z)}{\beta}\Big) \\ & -\frac{\pi}{8\beta^3 z}\Bigg(\coth\frac{\pi(\Delta t-2z)}{\beta}\mathrm{csch}^2\frac{\pi(\Delta t-2z)}{\beta} \\ & \qquad -\coth\frac{\pi(\Delta t+2z)}{\beta}\mathrm{csch}^2\frac{\pi(\Delta t+2z)}{\beta}\Bigg)\,, \end{aligned} \tag{20}$$

$$\begin{aligned} F^z_{\beta b}(\Delta t, z) \;=\; & -\frac{1}{32\pi\beta z^3}\Big(\coth\frac{\pi(\Delta t-2z)}{\beta} - \coth\frac{\pi(\Delta t+2z)}{\beta}\Big) \\ & +\frac{1}{16\beta^2 z^2}\Big(\mathrm{csch}^2\frac{\pi(\Delta t-2z)}{\beta} + \mathrm{csch}^2\frac{\pi(\Delta t+2z)}{\beta}\Big) \end{aligned} \tag{21}$$

Clearly, $F_{\beta m}$ is the electric field two-point function at finite temperature in Minkowski space while $F^x_{\beta b}$ and $F^z_{\beta b}$ are the correction induced by the presence of the boundary. With the electric field two-point function given, the velocity dispersion in the x-direction can be calculated out to be

$$\begin{aligned} \langle \Delta v_x^2\rangle \;=\; & \frac{e^2}{m^2}\Bigg\{\Big(\frac{1}{9\beta} - \frac{1}{3\pi^2 t^2} + \frac{\mathrm{csch}^2\frac{\pi t}{\beta}}{3\beta^2}\Big) - \frac{1}{16\pi^2 z^2}\ln\Bigg(\frac{\sinh\frac{\pi(t+2z)}{\beta}\sinh\frac{\pi(2z-t)}{\beta}}{\sinh^2\frac{2\pi z}{\beta}}\Bigg) \\ & -\frac{1}{4\pi\beta z}\coth\frac{2\pi z}{\beta}\;\mathrm{csch}\frac{\pi(t-2z)}{\beta}\;\mathrm{csch}\frac{\pi(t+2z)}{\beta}\;\sinh^2\frac{\pi t}{\beta} \\ & +\frac{\beta}{128\pi^3 z^3}\Big(g_\beta(t,z) - g_\beta(t,-z)\Big)\Bigg\}\,. \end{aligned} \tag{22}$$

Here we have introduced a new function $g_\beta(t,z)$, which is defined by

$$g_\beta(t,z) = \mathrm{PolyLog}[\,2, e^{\frac{2\pi(t-2z)}{\beta}}\,] + 2\mathrm{PolyLog}[\,2, e^{\frac{4\pi z}{\beta}}\,] + \mathrm{PolyLog}[\,2, e^{\frac{-2\pi(t+2z)}{\beta}}\,]\,, \tag{23}$$

where the polylogarithm functions, PolyLog[n, z,] are given by

$$\mathrm{PolyLog}[\,n, z\,] = \sum_{k=1}^{\infty}\frac{z^k}{k^n} \equiv PL[\,n, z\,]\,. \tag{24}$$

The velocity dispersion in the z-direction is given by

$$\langle \Delta v_z^2 \rangle \;=\; \frac{e^2}{m^2}\Biggl\{\Bigl(\frac{1}{9\beta}-\frac{1}{3\pi^2t^2}+\frac{\mathrm{csch}^2\frac{\pi t}{\beta}}{3\beta^2}\Bigr)-\frac{1}{8\pi^2z^2}\ln\Bigl(\frac{\sinh\frac{\pi(t+2z)}{\beta}\sinh\frac{\pi(2z-t)}{\beta}}{\sinh^2\frac{2\pi z}{\beta}}\Bigr)$$
$$+\frac{\beta}{64\pi^3z^3}\Bigl(g_\beta(t,z)-g_\beta(t,-z)\Bigr)\Biggr\}\,. \tag{25}$$

The mean squared fluctuations in both the transverse and longitudinal directions are evaluated to be

$$\langle \Delta x^2\rangle \;=\; \frac{e^2}{m^2}\Biggl(\frac{t^3}{8\pi\beta z^2}-\frac{t^2}{8\pi\beta z}\coth\frac{2\pi z}{\beta}-\frac{t^2}{32\pi^2z^2}\ln\Bigl[4\sinh^2\frac{2\pi t}{\beta}\Bigr]-\frac{t}{2\pi\beta}$$
$$+\frac{t}{8\pi^2z}\ln\Bigl[\mathrm{csch}\frac{\pi(t-2z)}{\beta}\sinh\frac{\pi(t+2z)}{\beta}\Bigr]\Biggr)+\frac{e^2}{m^2}\Biggl(\frac{\beta(t^2-8z^2)}{128\pi^3z^3}\mathrm{PL}[2,e^{\frac{-2\pi(t+2z)}{\beta}}]$$
$$+\frac{\beta(t^2-8z^2)}{128\pi^3z^3}\mathrm{PL}[2,e^{\frac{4\pi z}{\beta}}]+\frac{\beta t(t+4z)}{128\pi^3z^3}\mathrm{PL}[2,e^{\frac{2\pi(t+2z)}{\beta}}]+\frac{\beta^2}{64\pi^4z^2}\mathrm{PL}[3,e^{\frac{4\pi z}{\beta}}]$$
$$+\frac{\beta^2(t-2z)}{128\pi^4z^3}\mathrm{PL}[3,e^{\frac{2\pi(t-2z)}{\beta}}]+\frac{\beta^3}{256\pi^5z^3}\Bigl[\mathrm{PL}[4,e^{\frac{-4\pi z}{\beta}}]+\mathrm{PL}[4,e^{\frac{2\pi(t+2z)}{\beta}}]\Bigr]$$
$$+(z\to -z)\Biggr)\,, \tag{26}$$

and

$$\langle \Delta z^2\rangle \;=\; \frac{e^2}{m^2}\Biggl(\frac{t^3}{8\pi\beta z^2}-\frac{t^2}{32\pi^2z^2}\ln\Bigl[4\sinh^2\frac{2\pi t}{\beta}\Bigr]\Biggr)+\frac{e^2}{m^2}\Biggl(\frac{\beta t^2}{64\pi^3z^3}\mathrm{PL}[2,e^{\frac{-2\pi(t+2z)}{\beta}}]$$
$$+\frac{\beta t^2}{64\pi^3z^3}\mathrm{PL}[2,e^{\frac{4\pi z}{\beta}}]+\frac{\beta t(t+4z)}{64\pi^3z^3}\mathrm{PL}[2,e^{\frac{2\pi(t+2z)}{\beta}}]+\frac{\beta^2}{32\pi^4z^2}\mathrm{PL}[3,e^{\frac{4\pi z}{\beta}}]$$
$$+\frac{\beta^2(t-2z)}{64\pi^4z^3}\mathrm{PL}[3,e^{\frac{2\pi(t-2z)}{\beta}}]+\frac{\beta^3}{128\pi^5z^3}\Bigl[\mathrm{PL}[4,e^{\frac{-4\pi z}{\beta}}]+\mathrm{PL}[4,e^{\frac{2\pi(t+2z)}{\beta}}]\Bigr]$$
$$+(z\to -z)\Biggr)\,. \tag{27}$$

Here $(z \to -z)$ stands for all the terms in the big brackets but with the sign of z flipped. In the high temperature limit $t \gg z \gg \beta$, the velocity and position dispersions of the test particle in the directions parallel to the plate are approximately given by,

$$\langle \Delta v_x^2\rangle = \langle \Delta v_y^2\rangle \approx \frac{e^2}{9m^2\beta^2}-\frac{e^2}{8\pi m^2\beta z}+\frac{e^2\beta}{128\pi m^2z^3}-\frac{e^2}{3\pi^2m^2t^2}\,, \tag{28}$$

and

$$\langle \Delta x^2\rangle = \langle \Delta y^2\rangle \approx \frac{e^2}{m^2}\left(\frac{t^2}{18\beta^2}-\frac{t^2}{16\pi z\beta}+\frac{t}{2\pi\beta}-\frac{1}{3\pi^2}\ln\frac{\pi t}{\beta}+\frac{1}{3\pi^2}\right)\,, \tag{29}$$

while for the direction normal to the plate, we have

$$\langle \Delta v_z^2 \rangle \approx \frac{e^2}{9m^2\beta^2} + \frac{e^2}{4\pi m^2 \beta z} + \frac{e^2\beta}{64\pi m^2 z^3} - \frac{e^2}{3\pi^2 m^2 t^2} \,, \tag{30}$$

$$\langle \Delta z^2 \rangle \approx \frac{e^2}{m^2}\left(\frac{t^2}{18\beta^2} + \frac{t^2}{8\pi z\beta} + \frac{2t}{\pi\beta} - \frac{1}{3\pi^2}\ln\frac{\pi t}{\beta} + \frac{1}{3\pi^2}\right) \,. \tag{31}$$

Let us that note that, for $z = 1\mu m$, the condition $z \gg \beta$ leads to the requirement that the temperature of the system be much larger only than 10^{-3} K. Hence, depending on the value of z, a very low temperature in experiment may be considered as high temperature for the random motion discussed here. A comparison of the above results with Eq. (11) and Eq. (14) reveals that the random motion driven by quantum field fluctuations at finite temperature is reinforced in the normal direction and weakened in the parallel directions by the presence of a reflecting plate, which modifies the quantum field fluctuations. It is easy to see that even with this enhancement the random motion in the normal direction driven by quantum fluctuations is still much less significant than that driven by thermal noise. This is expected since when the temperature is high, the random motion should be dominated by thermal noise.

When the temperature of the system is very low, i.e., when $\beta \gg t$ and $\beta \gg z$, in the x-direction, the dispersions of the test particle are approximated as follows

$$\langle \Delta v_x^2 \rangle \approx \frac{e^2}{\pi^2 m^2}\left[\frac{t}{64z^3}\ln\left(\frac{2z+t}{2z-t}\right)^2 - \frac{t^2}{8z^2(t^2-4z^2)}\right] + \frac{32e^2\pi^4}{945m^2}\frac{t^2z^2}{\beta^6} \,, \tag{32}$$

$$\begin{aligned}\langle \Delta x^2 \rangle \;\approx\; & \frac{e^2}{\pi^2 m^2}\left[-\frac{t^2}{24z^2} + \frac{t^3}{192z^3}\ln\left(\frac{t+2z}{t-2z}\right)^2 - \frac{1}{6}\ln\left(\frac{t^2-4z^2}{4z^2}\right)\right] \\ & + \frac{8\pi^4 e^2}{945m^2}\frac{t^4z^2}{\beta^6} \,, \end{aligned} \tag{33}$$

and in the z-direction as follows

$$\langle \Delta v_z^2 \rangle \;\approx\; \frac{e^2}{\pi^2 m^2}\,\frac{t}{32z^3}\ln\left(\frac{2z+t}{2z-t}\right)^2 + \frac{64e^2\pi^4}{945m^2}\frac{t^2z^2}{\beta^6} \,, \tag{34}$$

$$\begin{aligned}\langle \Delta z^2 \rangle \;\approx\; & \frac{e^2}{\pi^2 m^2}\left[\frac{t^2}{24z^2} + \frac{t^3}{96z^3}\ln\left(\frac{t+2z}{t-2z}\right)^2 + \frac{1}{6}\ln\left(\frac{t^2-4z^2}{4z^2}\right)\right] \\ & + \frac{\pi^2 t^4}{90\beta^4} - \frac{2\pi^4(5t^6+18t^4z^2)}{8505\beta^6} \,, \end{aligned} \tag{35}$$

The β independent terms in all the above expressions result from the Brownian motion driven just by the quantum vacuum fluctuations, while β dependent terms represent the temperature corrections. When $\beta \to \infty$, the above results reduces to those given

in Ref. [4] for the Brownian motion in vacuum. Clearly, in the low temperature limit, the Brownian motion is dominated by the quantum vacuum fluctuations and the temperature corrections are higher order and thus negligible. It is worth noting that, depending on the value of the initial distance of the test particle from the plate, the temperature T may have to be extremely low in order for the low temperature condition $\beta \gg z$ to be obeyed. For example, for $z = 1\mu m$, the temperature T must be lower than 10^{-3} K. Therefore, in reality, we are more likely to face the high temperature limit, i.e., when $t \gg \beta$ and $z \gg \beta$ are satisfied.

However, if the system temperature is so low such that $\beta \gg t \gg z$ holds, then the random motion driven by quantum field fluctuations could become much more significant than the thermal random motion. For example, in this limit, the velocity dispersion of the charged test particle in the z-direction can be estimated as

$$\langle \Delta v_z^2 \rangle \approx \frac{e^2}{4\pi^2 m^2}\frac{1}{z^2} + \frac{64 e^2 \pi^4}{945 m^2}\left(\frac{t}{\beta}\right)^2\left(\frac{z}{\beta}\right)^2\frac{1}{\beta^2} + \frac{e^2}{3\pi^2 m^2}\frac{1}{t^2} \,. \tag{36}$$

Clearly the first term represents the contribution of quantum vacuum fluctuations, while the second β dependent term is the correction induced by system temperature being non-zero. With is result, it follows that in this case the ratio of the velocity dispersion due to the random motion driven by quantum fluctuations to that driven by thermal noise at the same temperature is

$$\frac{\Delta v_{qm}}{\Delta v_{th}} = \left(\frac{\alpha}{\pi}\right)^{1/2}\left(\frac{\beta}{z}\right) \,. \tag{37}$$

This demonstrates that in the low temperature the random motion of quantum origin can be orders of magnitude much more significant than the thermal random motion and thus the quantum fluctuations are the dominant driving source of the random of the test particles at low temperature. To experimentally verify the dominance of the random motion of the quantum origin over that of thermal origin, one needs to cool the system to a significantly low temperature, for example, for $z \simeq 10^2\mu m$, the system temperature, T has to be less than 0.1 K. The smaller the value of z, the lower the temperature T has to be.

In conclusion, we have been concerned with an interesting problem of the random motion of charged test particles driven by quantum electromagnetic field fluctuations at finite temperature. Here, the random motion is caused by ever-existing quantum electromagnetic fluctuations of a thermal bath of photons. A very interesting feature of the random motion discussed in the present paper, in contrast to that driven by thermal noise, is that no dissipation is needed for the velocity dispersion of the test particle to be bounded at later times. Our calculations also show that generally the random motion driven by quantum fluctuations is one order of magnitude less significant than that driven by thermal noise and it could be strengthened if the quantum field fluctuations are to be modified by the presence of a reflecting plane boundary. In particular, in the case with a reflecting plane boundary, the random motion of quantum origin in the direction normal to the boundary could become

orders of magnitude more significant than that of thermal origin, when the system temperature is low.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grants No.10375023 and No.10575035, the Program for NCET (No.04-0784), the Key Project of Chinese Ministry of Education (No.205110) and the Key Project of Hunan Provincial Education Department (No. 04A030)

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